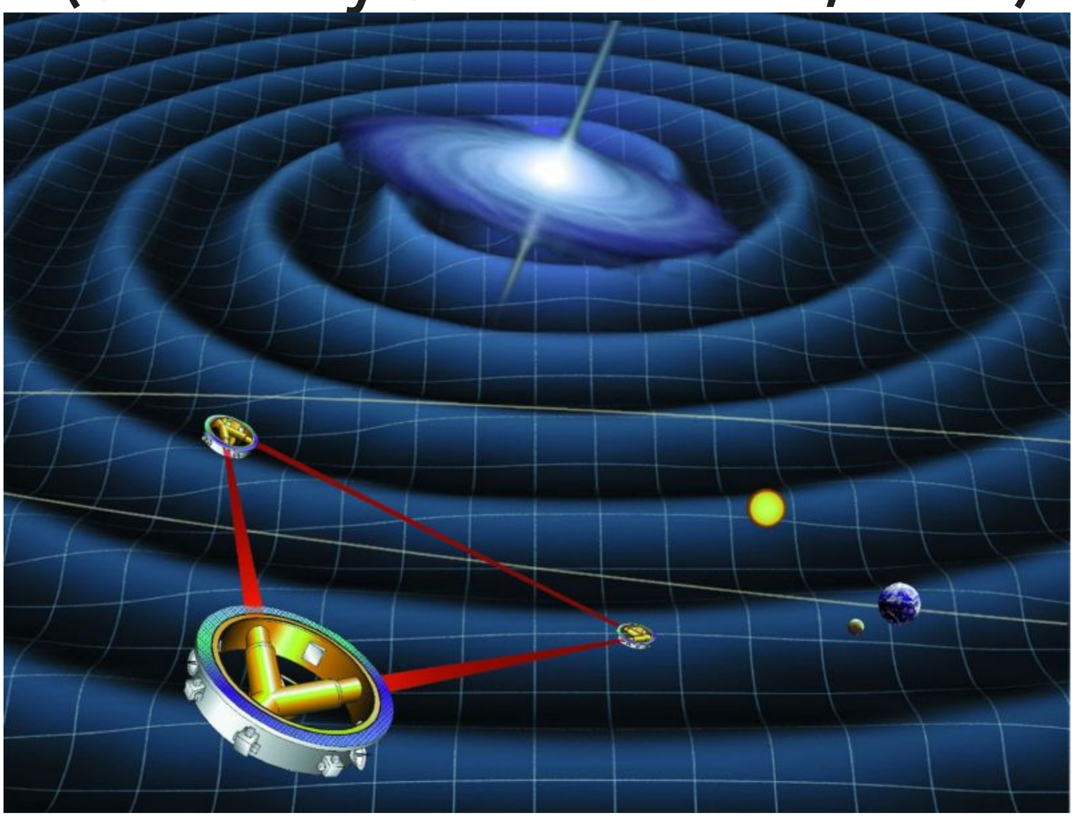
Online Student Manual

Hints, Tips, and Short Answers to Selected Problems in

Moore, *A General Relativity Workbook,*
(University Science Books, 2013)



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Preface

This study guide is meant to help both those people who are studying *A General Relativity Workbook* on their own as well as students who are using the book in a formal university course. The *GRW* textbook was *designed* to serve as a textbook for such a formal course, and I believe that studying the subject in a course setting with a knowledgeable instructor who can give you focused and detailed feedback still represents the best way to learn the material by far. However, I recognize that for many reasons, this is not practical for a number of readers, and even students taking such a course can benefit from some extra help.

The goal of this manual is, therefore, to provide to those studying the book (in any setting) some extra feedback that will help keep them on track and reassure them that they are learning the material correctly, without providing so much information that students taking the course for credit could use this guide to cheat and thereby short-cut the learning process. This is somewhat of a balancing act. You will not find in this guide *complete* solutions to either exercises or homework problems. Rather, you will find hints, short answers, a few complete solutions to problems *not* in the book, and other guidance that should give *everyone* useful feedback and direction without providing short cuts around the hard work that is essential to the learning process. Indeed, since the goal of the textbook is that everyone using it becomes both proficient and confident with the material, I hope that professors using the textbook in formal classes will find that this guide gives useful (not harmful) feedback to their students, saving professors the effort of explaining some basic things and allowing more class time for broader exploration of the subject.

Please note that almost all of the exercises and many of the homework problems in the textbook *already* provide answers that will give you at least some feedback. Getting the right answer using incorrect methods is generally more difficult with this material than it would be in an introductory course (and I will try to offer some guidance when this is not the case), so if you arrive at the right result, chances are pretty good that you are doing things correctly. At the same time, you should refer to this guide only *after* you have tried to work out the problem on your own.

Also, please note that while I provide short numerical or symbolic answers to some of the problems, the answers are not equivalent to *solutions*, which (whether you are a student in a formal course or not) should include steps of algebra and/or reasoning. The provided answers simply help you determine whether you have the solution right. (I have *not* provided answers to such problems where the solution is basically equivalent to the answer.)

In what follows, three digit numbers (such as 5.4.1) refer to exercises in the boxes, numbers preceded by P (such as P12.7) refer to homework problems, and numbers preceded by S (such as S12.1) refer to supplemental problems not in the book. Full solutions to any S problems appear at the end of this manual. (Note: There is only one S problems in the manual so far, but I hope to add some more soon.)

Please feel free to send me feedback about this guide, particularly error notices and constructive comments about how I might improve it for readers like yourself. Also, if you find that the information in this guide is insufficient to help you puzzle out the solution to a problem (given a sufficient background and plenty of hard work), I would like to hear about it. Send me an email at tmoore@pomona.edu.

Chapter 1

P1.2 (b) *Answer:* 2.7×10^{-15}

P1.3 (a) *Answer:* 4.9×10^{-16} m

P1.5 *Hint:* You should find that if the frame were falling just above the earth's surface (6380 km from the earth's center), the relative acceleration of ball B relative to ball A at the center would be 6.8×10^{-5} m/s^2 upward.

Chapter 2

2.1.1 *Hint:* Note that a free object at rest in frame S' will move at a constant velocity in S if S' does.

2.8.1 *Hint:* You need to use the Lorentz transformation equation that involves Δt, Δx, and $\Delta x'$.

2.9.1 *Hint:* Note that the answer is given in equation 2.6.

P2.1 *Partial answer:* Worldline *A* describes a particle that is at $x = 0$ at time $t = 0$ and is moving at a constant speed equal to the speed of light in the $-x$ direction. Worldline *F* is impossible (be sure to explain why).

P2.2 (c) *Partial answer:* Event *C* occurs at $x = 0$, $t = 10$ Tm.

P2.3 (c) *Answer:* In the spaceship, the elapsed time is 240 Gm.

P2.4 *Partial answer*: The nap lasts 3.0 Tm. Note that event *B* happens at $\Delta t = 1.25$ Tm and $\Delta x = 0.75$ Tm in the solar system frame.

P2.5 (b) *Answer:* $\Delta t' = 75$ m. (c) The speed of *S* is 3/5 relative to S'.

P2.6 (a) *Partial answer:* $\Delta x' \equiv x'_B - x'_A = -75$ m (negative because *A* occurs at the train's *front*).

P2.7 *Answer:* The front light flashes first in the train frame.

P2.8 (b) *Answer:* 300 m.

P2.9 (b) *Answer:* About 9.0 Tm.

P2.10 *Hint:* Consider three events: *A* (the light flash leaves the clock's left mirror), *B* (the flash bounces off of the clock's right mirror), and *C* (the flash returns to the left mirror). Show that $\Delta t_{BA} = t_B - t_A = L_S/(1-\beta)$, where L_S is the contracted length of the light clock in the ground frame.

P2.11 *Answer:* 37°

Chapter 3

3.5.1 *Hint:* Start by calculating the dot product of the vectors given in equation 3.42.

P3.1 (b) *Answer:* $u^t = \cosh(g\tau)$ (e) *Partial answer:* $v = gt/\sqrt{1+(gt)^2}$.

P3.2 (c) *Answer:* $u^t = e^{g\tau}$. (e) *Partial answer:* $x(t) - x(0) = \sqrt{(gt+1)^2 - 1} - \tan^{-1}\sqrt{(gt+1)^2 - 1}$.

P3.4 *Answer:* 30 MeV.

P3.5(b) *Hint:* The photons' energies are $m/(1+v)$ and $m/(1-v)$.

P3.6 (a) *Answer:* $m = 0.16M$ (b) *Answer:* $m = 0.00066M$.

P3.7 *Answer:* $\lambda = (1+v)^{1/2}(1-v)^{-1/2}\lambda_0$.

P3.8 *Hint:* In the CM frame, show that the photon's momentum would have to be zero, which is impossible (be sure that you can explain *why* that is impossible).

P3.9 *Answer:* $7m$.

P3.10 *Answer:* $4m$.

Chapter 4

4.1.1 The $\mu = t$ sum in the first case is $\delta^t{}_\nu A^\nu = \delta^t{}_t A^t + \delta^t{}_x A^x + \delta^t{}_y A^y + \delta^t{}_z A^z = 1 \cdot A^t + 0 \cdot A^x + 0 \cdot A^y + 0 \cdot A^z = A^t$.

4.3.1 *Hint*: Remember that $\vec{F} = d\vec{p}/dt$ but also that $\gamma = dt/d\tau = u^t$ in this case.

4.3.2 *Hint:* Remember that $\varepsilon_0 \mu_0 = 1$ in GR units.

4.5.1 *Hint:* Three of the five equations are OK.

4.5.2 *Hint:* Three of the six equations are OK.

P4.1 *Hint:* Two of the eight equations are correct.

P4.2 *Hint:* Two of the five equations are correct.

P4.5 *Hint:* Write out the implied sum.

P4.7 *Hint:* In interpreting the result, consider the work-energy theorem.

P4.11 *Hint:* Start with $\boldsymbol{u} \cdot \boldsymbol{u} = -1$.

S4.1 Use index notation to show that $\boldsymbol{A} \cdot \boldsymbol{B} \equiv \eta_{\alpha\beta} A^\alpha B^\beta$ has the same numerical value in all IRFs.

Chapter 5

5.1.1 *Hint:* Equation 5.18 predicts a vector pointing to the upper right, but $\Delta\boldsymbol{s}$ actually points to the upper left.

5.2.1 *Hint:* Argue that if $(\ldots)dx'^{\mu}dx'^{\nu} = 0$ for arbitrary displacements, then $(\ldots) = 0$.

5.3.3 *Hint:* In interpreting the results, note that the basis vectors are not generally perpendicular.

5.3.4 (b) *Hint:* Note that a vector purely in the $\boldsymbol{e}_p$ direction will generally have a nonzero y component.

5.6.1 *Partial answer:* $g_{\phi\phi} = R^2\cos^2\theta$.

P5.1 (a) *Partial answer:* $\dfrac{\partial x}{\partial r} = \cos\theta$, $\dfrac{\partial x}{\partial \theta} = -r\sin\theta$ (b) *Partial answer:* $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial \theta}{\partial x} = -\dfrac{y}{r}$

P5.3 (b) *Partial answer:* $v^r = v^2 t/r$.

P5.4 (b) *Partial answer:* a^q at the point in question has a value of –0.445 s^{-2}. (e) $|\boldsymbol{e}_q| = 1/bq$.

P5.5 (b) *Partial answer:* $g_{uw} = bA\cos(bu)$. (c) *Partial answer:* $v^w = -vbA\cos(bvt)$.
(d) *Hint:* Note that $u = x = vt$ at the object's position at time t. (e) *Hint:* $a^w = 0$ but $dv^w/dt \neq 0$

P5.6 *Hint:* Note that we can write θ in figure 5.5 as r/R.

P5.7 *Partial answer:* $g_{rr} = 1 + 4b^2r^2$.

Chapter 6

6.5.2 *Hint:* Remember that multiplication of factors in any given term is commutative. If you rearrange partial-derivative factors, two of the five factors collapse to a Kronecker delta.

6.5.3 *Hint:* Again, when re-arranged, two of the three partial-derivative terms collapse to a Kronecker delta.

P6.2 $v^{\mu}\partial_{\mu}\Phi = by$ in both coordinate systems.

P6.3 *Hint:* All partial derivative terms collapse to a pair of Kronecker deltas.

P6.4 (a) We *cannot* add these vectors (why not)?

P6.5 *Hint:* The answer is not 1. It might help to begin writing out the terms in the implied sum.

P6.6 (a) *Hint:* First exchange the indices, then note that multiplication of the partial derivatives is commutative.
(c) *Hint:* Prove that the quantity is equal to negative itself. (d) *Hint:* You can use the result of part c.
(e) *Partial answer*: Anti-symmetric tensors have 6 independent components

P6.7 (a) *Hint:* Note that $\dfrac{dA'^{\mu}}{d\tau} = \dfrac{d}{d\tau}\left(\dfrac{\partial x'^{\mu}}{\partial x^{\nu}}A^{\nu}\right) = \dfrac{dx^{\alpha}}{d\tau}\dfrac{\partial}{\partial x^{\alpha}}\left(\dfrac{\partial x'^{\mu}}{\partial x^{\nu}}A^{\nu}\right)$ by the chain rule. Take it from there.

P6.8 *Answer:* Only D is possible.

P6.9 (a) *Partial answer:* $\partial^{\mu}\partial_{\mu}\Phi = 4\pi G\rho$.

P6.10 (a) *Partial answer:* The sign of any component with exactly one t index gets reversed.
(c) You should get the z component of Faraday's law.

Chapter 7

7.6.2 *Partial answer:* The equation where $\mu = \nu = t$ yields $F^{tt} = 0$.

7.7.1 *Hint:* You should find that you have six terms that cancel in pairs.

P7.3 (c) *Answer:* $\partial_{\nu}\partial^{\nu}A^{\mu} = -4\pi kJ^{\mu}$.

P7.5 *Hint:* Expand out the implied sum and multiply both sides by $d\tau/dt$.

P7.7 *Answer:* $2(B^2 - E^2)$.

P7.8 (b) *Partial answer:* $T^{\mu\nu} = (1/4\pi k)(F^{\mu\alpha}\eta_{\alpha\beta}F^{\nu\beta} - \ldots)$. Argue that there is only one other possible distinct combination of two $\boldsymbol{F}$ tensors and a metric tensor that is a second-rank tensor and can therefore possibly be the omitted term.

Chapter 8

8.2.1 *Hint:* Look up "integrating by parts" if you don't know what that means. Also note that $f^\alpha(1) = f^\alpha(0) = 0$.

8.3.1 *Hint:* Since $f^\alpha(\sigma)$ can be any arbitrary function, consider a function that has a narrow peak around a particular value of σ but is zero elsewhere.

8.4.3 *Hint:* Many terms cancel if you use $p = bs$ (see equation 8.36).

P8.3 (e) *Hint:* You can easily integrate by substitution: let $u \equiv r^2 - c^2$.

P8.4 (a) *Hint:* $\frac{d}{d\tau} g_{\alpha\beta} = \frac{dx^\gamma}{d\tau}\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}$, because the metric depends on the coordinates and the coordinates along the worldline depend on τ. (b) Rename some indices and note that multiplication is commutative.

P8.6 (e) *Answer:* $x(\tau) = -2a\ln\left[\cos\left(\frac{\tau}{2a}\right)\right]$, $t(\tau) = 2a\tan\left(\frac{\tau}{2a}\right)$.

P8.7 (b) Remember that the metric implicitly depends on the geodesic's pathlength s as we move along the geodesic because the metric depends on the coordinate r and r depends on s.

Chapter 9

9.1.2 *Hint:* The answer is on the order of 10^{-4} m.

9.2.2 *Hint:* You should find that factors of $(1 - 2GM/r)$ in the numerator and denominator cancel out.

9.4.2 *Hint:* you should find that $-\boldsymbol{p} \cdot \boldsymbol{u}_R = E_E(1 - gh)$.

P9.1 (a) *Answer:* 0.053. (b) *Answer:* 0.083.

P9.2 *Hint:* You should find the results to be consistent.

P9.3 *Hint:* Argue that the metric implies that $ds = \pm R\, d\theta$.

P9.4 *Hint:* You should find that the integral you want to look up is $\int \frac{du}{u^2\sqrt{1-u}}$. WolframAlpha will give you a useful result in this case.

P9.5 *Hint:* the answer is a bit more than 25% larger than what you would naively expect.

P9.6 *Partial answer:* $p^r = E\sqrt{1 - 2GM/R}$ and this is *not* equal to p^t. Also note that the photon is moving radially by hypothesis: what does this mean about p^θ and p^ϕ?

P9.7 (a) *Answer:* Clocks at *rest* anywhere (explain why). (c) *Partial answer:* Circumference is greater than $2\pi r$.

P9.8 (b) *Hint:* Drop a term involving gzv^2. (c) *Hint:* The Lagrangian will be $L = gz - \frac{1}{2}\dot{z}^2$.

Chapter 10

10.2.1 *Hint:* Remember that $\sin\theta = 1$ on the equatorial plane.

10.4.3 *Answer:* $\ell \approx 3.46GM$ (you should be able to find the *exact* value, which involves a square root).

10.5.1 *Hint:* Remember that the Schwarzschild metric is diagonal.

10.5.2 *Hint:* Remember that $\sin\theta = 1$ on the equatorial plane.

10.7.1 *Answer:* $\Delta e \approx 0.057$ (you should be able to find the *exact* value, which involves a square root).

P10.2 *Answer:* $13.6\,GM$.

P10.3 *Answer:* The ratio is 1.58.

P10.4 (a) *Partial answer:* dr/dt goes to zero in this limit: the falling object will appear to "freeze."
(b) *Partial answer:* $dr/dt = -(1 - 2GM/r)(1 - 2GM/r_0)^{-1/2}\sqrt{2GM/r - 2GM/r_0}$.

P10.5 *Hint:* Argue that the time measured on both legs must be the same. If you ask WolframAlpha to solve the original integral (without the suggested substitution) you get something that is very tricky to interpret: you can't just throw away the imaginary part that Wα gives you. With the substitution, you should get something like a constant times $\sin^{-1}\sqrt{u} - \sqrt{u(1-u)}$ to evaluate at the endpoints.

Chapter 10 (continued)

P10.6 (b) *Answer:* 8.2 ms.

P10.8 *Answer:* $\ell = 4GM$.

P10.9 (b) *Partial answer:* $\tilde{E} = -0.0429$ (c) *Answer:* 13.6 min.

P10.10 *Hint:* The ratio between ω and Ω will involve a power of $(1 - 3GM/r)^{1/2}$.

P10.11 $r_c = GM(\ell/GM)^2$.

P10.12 *Partial answer:* speed as inferred by spaceship > speed for observer at r > speed for observer at infinity. The first can be greater than 1 (but is only an *inferred* speed).

P10.13 (b) *Answer:* The photon's speed according to the stationary observer is 1. (c) *Hint:* Consider whose clocks are being used to calculate the speed in each case.

P10.14 (e) *Answer:* The comet's speed according to the observer is $(\ell/R)\sqrt{1 - 2GM/R}$. (f) *Partial answer:* The points of closest approach is a bit closer than Newtonian physics would predict.

P10.15 (b) *Answer:* The necessary speed is about 0.956.

Chapter 11

11.4.1 *Hint:* You will get 41 as/century, not the observed 43 as/century, mostly because the mean orbital radius is not the same as a circular orbit radius.

11.6.1 *Hint:* Show that $\alpha \approx \sqrt{2GM/r}$ in this limit.

P11.2 *Answer:* 1560 as/century

P11.3 *Partial result:* 46°/s.

P11.4 *Hint:* The result is very close to the shift observed.

P11.7 *Answer:* $z(r) = \pm R\cosh(r/R) + C$.

P11.9 *Hint:* You should find that we must have $1 + \left(\frac{dz}{dr}\right)^2 = \frac{r^2}{r^2 - b^2}$. Solve for dz/dr and integrate.

Chapter 12

12.4.1 *Hint:* Start by dividing both sides of equation 12.20 by dt^2.

12.6.1 *Hint:* If the vectors point in the ϕ, $-\theta$, and r directions, respectively, then only their ϕ, θ, and r Schwarzschild components will be nonzero, respectively. This will ensure that they are orthogonal. So make sure that each has unit length.

P12.1 *Answer:* No.

P12.2 *Answer:* $\sqrt{27}\,GM$.

P12.3 *Hint:* You should find that $v_{x,\text{obs}} = \frac{\boldsymbol{o}_x \cdot \boldsymbol{p}}{-\boldsymbol{o}_z \cdot \boldsymbol{p}} = \frac{b}{r}\sqrt{1 - \frac{2GM}{r}}$.

P12.4 *Partial answer:* at $2.5GM$, $\psi_c = 112°$.

P12.6 (c) *Hint:* See figure 8.5.

P12.7 (b) *Hint:* Because $\boldsymbol{o}_z$ points in the radial direction spatially, its θ and ϕ components must be zero. Requiring that $\boldsymbol{o}_t \cdot \boldsymbol{o}_z = 0$ and $\boldsymbol{o}_z \cdot \boldsymbol{o}_z = 1$ puts two constraints on the other two components of $\boldsymbol{o}_z$ that allows them to be determined. You can do something similar with the other vectors. (c) *Hint: Partial answer*:

$$\sin\psi_c = \frac{(\sqrt{27}\,GM/r)(1 - 2GM/r)}{1 \pm \sqrt{2GM/r}\sqrt{1 - (27G^2M^2/r^2)(1 - 2GM/r)}}$$

The positive sign refers to an outgoing photon. Argue carefully that nearly critical photons from infinity will be outgoing when the observer's radius is $r > 3GM$ and incoming when $r < 3GM$ (drawing a picture might help). (d) *Partial answer*: Redshifted.

P12.8 *Hint:* Calculate the components $U^t = -\boldsymbol{o}_t \cdot \boldsymbol{u}$, $U^x = \boldsymbol{o}_x \cdot \boldsymbol{u}$, etc. of the orbiting object's four-velocity in the falling observer's frame. Note that $v_x = U^x/U^t$, etc. You should find that $v = \sqrt{3GM/r}$.

P12.9 *Partial answer:* $(\boldsymbol{o}_t)^\mu = (1 - 3GM/r)^{-1/2}[1, 0, 0, \sqrt{GM/r^3}]$.

S12.1 **(a)** Consider a particle orbiting in the equatorial plane of a Schwarzschild spacetime at $r = 6GM$. The definition of the Schwarzschild r coordinate implies that if the particle moves through an angular displacement of $d\phi$ in a certain coordinate time dt, the physical distance the particle moves is $r\,d\phi$. Therefore, an observer at infinity (whose clock measures time dt) will conclude that the particle's speed is $r\,d\phi/dt$. Show that the particle's Schwarzschild coordinate speed (so defined) is $v_\infty = \sqrt{1/6}$.

(b) Use the Schwarzschild metric to show that for this particle, $dt/d\tau = \sqrt{2}$, and from that, determine this particle's "proper speed" $v_P \equiv r\,d\phi/d\tau$ (the distance it covers divided by its own proper time).

(c) Consider an observer that is freely falling from rest at infinity. Just as the observer passes $r = 6GM$, the orbiting particle zips through the falling observer's orthonormal frame. Use the results from problem P12.7 to evaluate the components and the magnitude of the particle's velocity $\vec{v}_F$ as measured in that frame. (*Hint*: First find the Schwarzschild components of the particle's four-velocity $\boldsymbol{u}_P$. Note also that $\vec{v}_F$ will have two nonzero components.)

Chapter 13

13.4.1 *Partial answer:* $A = \varepsilon/2$.

13.7.1 *Hint:* Keep very careful track of the possible sign cases (you might want to do the two cases separately).

13.7.2 *Hint:* Multiply the expression given for x by x.

P13.2 (a) *Answers:* $1.281\theta_E, -0.781\theta_E$.

P13.3 (b) *Answer:* 1.22×10^{12} solar masses.

P13.4 (a) *Partial answer:* $\theta_- \approx -\theta_E^2/\beta$. (b) *Note:* The problem statement was meant to imply that you should keep terms of order $(\theta_E/\beta)^2$ but not of order $(\theta_E/\beta)^4$ in your calculation. But you should find that the terms of order $(\theta_E/\beta)^2$ cancel out. Specify the final results ignoring terms of order $(\theta_E/\beta)^4$, even though these terms are the leading correction terms remaining after the cancellation.

P13.5 (a) *Note:* Don't fret about making the estimate for D_S: choose something obvious and easy to calculate. You should get a time on the order of some tens of days.

P13.6 (c) *Hint:* Use trial and error to get a good match for the graph. *Partial answer:* $u_0 \approx 0.057$.

P13.8 (c) *Answer:* 660 d.

P13.9 (d) *Answer:* about 700,000 times.

Chapter 14

14.2.1 *Note:* WolframAlpha gives usable results for either integral.

P14.2 (c) *Answer:* 6.7 s

P14.4 *Hint:* You should find that the cones become more narrow as you approach the horizon, and then suddenly flip to face $r = 0$ and become extremely broad, though they narrow as you approach $r = 0$.

P14.5 *Hint:* $u^z_{\text{obs}} = \dfrac{\sqrt{2GM/R}}{\sqrt{1 - 2GM/R}}$ is one of the object's four-velocity components according to the fixed observer.

P14.6 *Hint:* Inside the event horizon, you should find that $\dfrac{dr}{dt} = -\left(\dfrac{2GM}{r} - 1\right)$ and $p^t = -E\left(\dfrac{2GM}{r} - 1\right)^{-1}$. You can find the first using $0 = ds^2$ for a photon.

Chapter 15

15.4.3 *Hint:* I recommend calculating each term term on the left separately, and *then* combining them. Some terms will cancel, leaving the right side of equation 15.22.

P15.2 *Hint:* To start things off, note that $dr/d\tau = -\sqrt{2GM/r}$, whose magnitude is greater than 1. But is r a radial distance inside the event horizon?

P15.3 *Hint:* WolframAlpha gives something useless for the original integral: use the recommended substitution. You should find that $\mathring{t}(r) = -r + \sqrt{8GMr} - 4GM\ln|1 + \sqrt{r/2GM}| + C$ for an in-going photon.

P15.4 *Answers:* No, and no.

P15.8 *Answer:* Observers at the front can see the tail at all times but will not see the tail end crushed.

P15.9 (b) Answer: $2.773\,GM$.

Chapter 16

16.1.1 *Hint:* Note that $dr/d\tau$ is negative because the particle is falling inward.

16.1.2 *Hint:* You should be able to use the binomial approximation to reduce the integral to $\Delta\tau = \int_0^\varepsilon \frac{\sqrt{2GM}}{\sqrt{\varepsilon - \rho}}\,d\rho$.

P16.2 (b) *Answer:* about 50 million bombs.

P16.3 *Hint:* less than 10^{-87} s.

P16.5 *Answer:* r needs to be larger than $2GM$ by about an eighth of a femtometer.

P16.6 *Hint:* The ratio $M(t)/M_0$ involves a simple fractional power of an appropriate binomial in t/t_0.

P16.9 *Hint:* The black hole's S/k_B is on the order of 19 orders of magnitude larger than that of ordinary matter.

Chapter 17

17.6.1 *Answer:* $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$, $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$.

17.7.1 *Hint:* There are four terms where the numbers are all the same and four where they are all different, and three in each of four groups where two numbers are the same.

17.7.2 *Hint:* two of the missing terms are $\mathbf{cag}_P$, $\mathbf{ba}\partial\mathbf{g}_P$.

P17.1 Answer: $\Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$, all others zero.

P17.3 (a) *Answer:* The only nonzero Christoffel symbols are $\Gamma^r_{\theta\theta} = -r$, $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = 1/r$.
(b) *Partial answer:* Gauss's law becomes

$$\frac{\partial E_r}{\partial r} + \frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z} + \frac{E_r}{r} = 4\pi k\rho$$

P17.5 (b) Answer: The only nonzero component of $\nabla_\mu A^\nu$ is $\nabla_p A^q = Cbq$.

P17.6 *Hints:* Because the metric is not diagonal, it is not easy to use the geodesic equation to evaluate Christoffel symbols (and for 2D metrics, this approach does not save much work anyway). To use equation 17.10, you will need to find the inverse metric, which (by the usual method for matrix inversion) is

$$g^{\mu\nu} = \frac{1}{\det(g)}\begin{bmatrix} g_{ww} & -g_{wu} \\ -g_{uw} & g_{uu} \end{bmatrix} = \begin{bmatrix} 1 & -bA\cos(bu) \\ -bA\cos(bu) & 1 + (bA)^2\cos^2(bu) \end{bmatrix} \quad \text{(after a bit of work)}$$

You should find that only the Γ^w_{uu} Christoffel symbol is nonzero.

P17.7 *Partial answer:* You should find that $a^r = GM/r^2$, but that this is not equal to $a \equiv \sqrt{\boldsymbol{a}\cdot\boldsymbol{a}}$.

P17.8 *Partial answer:* $\nabla_t v^r = +(GM/r^2)(1 - 2GM/r)^2$.

Chapter 18

18.1.2 *Hint:* Note that $x = y = 0$ and $z = r$ in our situation.

P18.1 *Answer:* About 26 ms.

P18.2 *Partial answer:* The field is *not* real in general relativity.

P18.3 *Hint:* Argue that the only possibly nonzero Riemann tensor component is $R^x{}_{xxx}$ and that this is zero.

P18.4 *Hint:* Since this is a metric for flat space, what had this component better be?

P18.5 *Answer:* The spacetime is *not* flat.

P18.6 (b) *Answer:* $R^t{}_{rtr} = \dfrac{2GM}{r^3}\left(1 - \dfrac{2GM}{r}\right)^{-1}$.

P18.7 *Hint:* Note that at the *origin* of a LIF that remains a LIF as time passes, all the Christoffel symbols *remain* zero as time passes.

P18.7 (b) *Hint:* Begin by renaming indices $\mu \leftrightarrow \nu$ in equation 18.38 and subtracting the two versions. Then you can rename indices to pull out a common factor of a^β. Remember that Christoffel symbols are symmetric in the lower two indices.

Chapter 19

19.2.3 *Hint:* You should have twelve terms that cancel in pairs.

19.3.2 *Hint:* You have two distinct cases (1) where the first index is the same as some other index, and (2) where two of the last three indices are the same. Consider these cases separately.

19.3.3 *Hint:* In a 2D space, there are four nonzero components, all of which are related by symmetry principles.

19.4.1 *Hint:* Again, expanding this out in a LIF yields twelve terms that cancel in pairs.

19.6.3 *Partial answer:* $R = 2/r^2$.

P19.1 *Hint:* Express $R^\alpha{}_{\beta\mu\nu}$ in terms of (say) $R_{\gamma\beta\mu\nu}$ and then contract over the α and β indices. Remember that only zero is equal to negative itself.

P19.2 *Hint:* Check your answer using the answer to problem P19.3. Also note that in spacetimes with less than four dimensions, equation 19.4*d* does not put any additional constraints on the components (explain why).

P19.3 *Hint:* Argue that the number of constraints implied by equation 19.4*d* is $n(n-1)(n-2)(n-3)/4!$ when we have $n \geq 4$.

P19.5 *Hint:* Semilog coordinates are a transformation of flat space (see problem P5.4).

P19.6 (b) This space is curved.

P19.8 (a) *Hint:* Note that $A^\mu{}_{\text{obs}} = \eta^{\mu\nu} A_{\nu,\text{obs}}$ because the metric in the observer's orthonormal coordinate system is $\eta_{\mu\nu}$ by definition. Note that $\boldsymbol{M} = \boldsymbol{AB}$ is a tensor if $\boldsymbol{A}$ and $\boldsymbol{B}$ are 4-vectors.

P19.9 *Partial answer:* The numerical result depends somewhat on what tension force you think a person can tolerate, but the person will not have time to suffer.

Chapter 20

20.1.1 *Hint*: Energy is conserved.

20.2.1 *Hint*: Show that one can express this in the form $[(\ldots)/A\Delta t]p^j$ where $(\ldots)$ is the number of particles moving through an area A that is perpendicular to the ith direction during a time interval Δt, and that one can do something analogous with a p^i behind the brackets instead of a p^j.

20.3.1 *Hint:* Note that all the quantities in the integrand involve the *squares* of the momentum components.

20.3.2 *Hint:* Can there be a physical distinction between the x, y, and z directions? Why or why not?

20.3.3 *Hint:* $N(|\vec{p}\,|)$ is the number density of particles having momentum magnitude $|\vec{p}\,|$, and p^t is the energy of a single particle with that momentum magnitude. Interpret what the rest of the formula means.

P20.1 *Hint:* You should find that T^{xx} has the same units but is about 360,000 times smaller in magnitude than T^{tt}.

P20.2 *Answer:* $\sqrt{[v^2]_{\text{avg}}} = 0.173$.

Chapter 20 (continued)

P20.3 *Partial answer:* $T^{tx} = \rho y \omega$.

P20.4 *Hint:* A fast way to the answer is to show that in this case, $T^{xx} + T^{yy} + T^{zz} = T^{tt}$.

P20.5 (a) *Hints*: First show that in a LIF, lowering the indices of the EM field tensor changes the signs of any term with one spatial and one time index, but leaves the signs of other terms alone. Then argue that there are only two possible distinct products of two factors of $\boldsymbol{F}$ and any number of factors of the metric that yield symmetric second-rank tensors. Find the right combination of these two distinct products that yield equation 20.32 in a LIF. In part (b), your result will involve $\vec{E} \times \vec{B}$.

P20.6 (a) *Hint:* One way to look at this is to treat $T^{\mu i}$ in a LIF as a vector $\vec{H}$ (with three spatial components indexed by i) that specifies in what direction and how fast something is flowing (in this case, the "something" is the μ component of four-momentum). How would you calculate the flux of $\vec{H}$ through a patch of area $A\vec{n}$? Then rewrite the vector notation in index form, noting that the four-vector $\boldsymbol{n}$ that in our LIF is perpendicular to a purely spatial patch has no time component.

P20.7 *Hint:* The pressure of this "fluid" will be negative.

P20.8 *Hints:* First, note that $A^{\mu}_{\text{obs}} = \eta^{\mu\nu} A_{\nu,\text{obs}}$, since the metric in the observer's orthonormal frame is $\eta^{\mu\nu}$ by definition. Second, consider that $\boldsymbol{M} = \boldsymbol{AB}$ (the tensor product of $\boldsymbol{A}$ and $\boldsymbol{B}$) where $\boldsymbol{A}$ and $\boldsymbol{B}$ are four-vectors, is a second-rank tensor. Since know how to calculate the components of $\boldsymbol{A}$ and $\boldsymbol{B}$, we can use this example tensor to figure out how to calculate the components of $\boldsymbol{M}$.

P20.9 *Answer:* $\alpha \geq -1$.

Chapter 21

21.2.2 *Hint:* Group an appropriate metric factor with the Riemann tensor in the last two terms to get a factor of the Ricci tensor. Then rename indices to combine those terms.

21.3.1 *Hint:* Write out the implied sum.

P21.2 *Hint:* Show that $G = g_{\mu\nu} G^{\mu\nu} = -R$.

P21.3 (b) *Partial answer:* $T^{tt} - \frac{1}{2} g^{tt} T = \frac{1}{2}(\rho_0 + 3p_0)$.

P21.4 *Partial answer:* Yes.

P21.5 *Hint:* Begin by giving the two spacetime coordinates abstract names, such as 0 and 1. You should find that $R_{\mu\nu} = 0$ means that the only possibly nonzero component of the Riemann tensor must also be zero.

P21.6 *Hint:* Begin by giving the three spacetime coordinates abstract names, such as 0, 1, and 2. Then determine and list the six independent Riemann tensor components (use the form where the all the indices are subscript). Then write the components of the Ricci tensor in terms of these components in a LIF. For example, you should find that $R_{00} = R_{0101} + R_{0202}$ in a LIF.

P21.7 *Hint:* First show (component by component) that $R^{\mu\nu} = g^{\mu\nu}/r^2$ in this particular case.

Chapter 22

22.2.1 *Hint:* You will ignore a second-order term with the form $b^{\mu\nu} h_{\mu\alpha}$.

22.3.1 *Hint:* Note that $\partial_\mu \eta_{\alpha\beta} = 0$.

22.5.1 *Hint:* Also remember that $\delta^{\mu}_{\mu} = 4$.

22.6.1 Hint: Note that $u^t = (1 - v^2)^{-1/2} \approx 1$ to first order in v.

P22.2 *Partial answer:* $\rho_g = -\Lambda/4\pi G$.

P22.3 *Hint:* Write $\vec{v} \cdot \vec{a}$ in index form and show that it is equal to negative itself.

P22.4 *Hint:* The conditions on $A_{\mu\nu}$ that you find should be satisfied if $A_{t\alpha} = A_{\alpha t} = A_{x\alpha} = A_{\alpha x} = 0$ and $A_{xx} = -A_{yy}$.

P22.5 *Partial answer:* $F_{xy} = -2GS(z^2 - x^2 - y^2)/r^5$. Remember also that F_{jk} is anti-symmetric.

P22.6 (c) *Partial answer:* $d^2x/dt^2 = -(2G\lambda/r)(x/r) + 4(2G\lambda V/r)v^z(x/r)$. Note that a moving wire with *charge* per unit length λ in classical electromagnetism would produce electric and magnetic fields $\vec{E} = (2k\lambda/r)\hat{r}$ and $\vec{B} = (2k\lambda/r)(\vec{V}\times\hat{r})$, respectively, where $k = 1/4\pi\varepsilon_0$ is the Coulomb constant (also remember that $\mu_0\varepsilon_0 = 1/c^2 = 1$ in GR units).

P22.7 (c) Partial answer: $d^2x/dt^2 = 16\pi G\sigma V v^z$. Note that if σ were a surface *charge* density, the electric and magnetic fields produced by each plate would be $\vec{E} = 2\pi k\sigma(\pm\hat{z})$ and $\vec{B} = 2\pi k\sigma[\vec{V}\times(\pm\hat{z})]$ above and below the plate, respectively, where $k = 1/4\pi\varepsilon_0$ is the Coulomb constant (also remember that $\mu_0\varepsilon_0 = 1/c^2 = 1$ in GR units).

Chapter 23

P23.1 (d) *Hint:* Write out the definition of $R_{\alpha\beta\mu\nu} = g_{\alpha\sigma}R^{\sigma}{}_{\beta\mu\nu}$ in terms of Christoffel symbols, and note that the only nonzero Christoffel symbols involve two t indices and one x index, and that the metric is diagonal.

P23.2 *Short answers:* (a) Larger r. (b) No. (c) No.

P23.3 (d) *Answer:* It adds a term $-\frac{1}{6}\Lambda(r^2 + \ell^2)$.

(For the other problems in this chapter, note that answers and quite complete directions are given in each problem.)

Chapter 24

P24.1 *Hint:* For simplicity, assume that the alien ship lies in the same plane as the two observatories and the sun's center (a good approximation if the ship is in front of the sun at all). *Answer:* 22.0 (±0.7) × 10^6 km.

P24.2 (c) *Answer:* 382 pc ± 25 (or so) pc (the uncertainty estimate depends on how you calculate it).

P24.3 *Answers:* (a) 8.5×10^{-11} times solar flux. (b) 1.6×10^{20} y.

P24.4 *Short answers:* (a) $v = 0.27$. (b) 3.57 Gy.

P24.5 *Partial answer:* $z = 3.13 \times 10^{-6}$.

P24.6 *Short answers:* (b) 17.1 Mly. (c) About 2 chances in a billion.

P24.7 *Hint*: You should be able to get "ChiSq" below 58. Try a halo central density near the upper limit.

Chapter 25

25.3.1 *Hint:* Note that $u^\mu g_{\nu\alpha} u^\nu = 0$ if and only if $\alpha = \nu = \mu = t$.

25.5.1 *Hint:* Make a table whose rows are the expressions for q and columns are q', q'', qq'', and $(q')^2 - qq''$.

P25.2 *Hint:* You should, of course, get the same possible expressions for q that we found before.

P25.3 *Hint:* Argue that for any geodesic *going through the origin*, $a^2q^2\sin^2\theta\frac{d\phi}{d\tau} = 0, a^2q^2\frac{d\theta}{d\tau} = 0$.

P25.4 *Partial answer*: $k = \pm 1/R^2$ or zero.

P25.5 You should find that the process requires that $1 = \left(1 + \left[\frac{dz}{dr}\right]^2\right)\left(1 + \left[\frac{r}{aR}\right]^2\right)$, which cannot be satisfied.

P25.8 *Short answer:* $2\pi^2a^3R^3$.

P25.9 (b) *Hint:* What happens as r/t goes to 1?

Chapter 26

26.7.1 *Partial answer:* if $\Omega_k > 1$ (meaning that $\Omega_m = 1 - \Omega_k < 1$), the universe is "open" (saddle-like), and will expand forever.

P26.1 *Partial answer:* if $\Omega_r < 1$, the universe will expand forever.

P26.2 *Partial answer:* a "Big Bang" is not possible if $\Omega_v > 1$.

P26.3 (b) *Partial answer:* $\rho_v = \frac{1}{2}\rho_{m0}$. (c) Partial answer: $\rho_v = 1.69 \times 10^{-21}$ kg/m^3. (e) *Partial answer:* $R = 18.8$ Mly.

Chapter 26 (continued)

P26.5 (c) *Hint:* This is a bit of a trick question.

P26.7 (a) *Partial answer:* age = t_0. (b) *Partial answer:* flat. (c) *Partial answer:* Radiation dominated.

P26.8 (g) *Answer:* $t(\psi) = (\Omega_m/2H_0)(\Omega_m - 1)^{-3/2}(\psi - \sin\psi)$ (i) *Partial answer:* 1540 Gy from Bang to Crunch.

P26.9 *Short answers:* (b) Age of universe = (time since $t = 0$) = 9.35 Gy. (d) R = 15 Gy.

Chapter 27

27.5.2 *Partial answer:* $a_4 = 5.16 \times 10^{-4}$.

P27.1 *Hint:* Between 3 and 4 times 13.75 Gy.

P27.2 *Short answer:* $d_A = a_e q_e$.

P27.3 *Hints:* Your results should look identical to figure 27.3*b* if you have done it right. Remember to plot a versus t, which you will have to calculate using a difference equation based on equation 27.3.

P27.4 *Hint:* You should find that $d_L \approx 52$ Gly at $z = 2$.

P27.5 *Short answer:* 20 Gly.

P27.7 (c) *Short answer:* 13.9 Gy (d) *Short answer:* 1,270,000 y. (f) *Partial answer and hints:* the light travel time is 6.95 Gy, but the present distance to the galaxy is significantly larger, and d_L is greater still.

Chapter 28

28.4.3 *Hint*: Note that at 1 MeV, electrons, positrons, photons, and three kinds of neutrinos are relativistic. Each neutrino has only one possible spin orientation.

P28.3 (a) *Short answer:* $k_B T$ with 4 families = 0.718 MeV. (b) *Hint:* Assume that the same fraction of neutrinos decays during the 200-s time interval. You should find that the ^{4}He ratio increases by about 3.9%.

P28.4 *Hint:* Your answer should be on the order of a few microseconds.

P28.6 *Partial answer:* The average neutrino mass must be smaller than about 1 eV.

Chapter 29

29.5.2 *Hint*: Including the correct value of g^* makes a difference of about a factor of 6, but this does not really change the order of magnitude of the transition time.

P29.1 *Short answer:* $\Omega_k = -0.486$.

P29.2 *Hint*: I had to use time steps of about $\Delta\eta = 0.0002$ to get a reasonably precise value of a_{PD}. I found that $\bar{t}_{PD} = 0.924$ Gy: you also should find $\bar{t}_0$ to three decimal places. You will find that our original estimate for $\Delta\phi$ was somewhat too large.

P29.3 *Partial answer:* the size is maximum when $z = 1.25$.

P29.4 *Hint:* Note that $\bar{t}_0 = 47$ Gy.

P29.5 *Short answer:* ~60 times. *Hint:* You will need to solve a transcendental equation: WolframAlpha can help.

P29.8 *Hint:* Place one point at the origin, the other point at a fixed comoving radial coordinate $\Delta\bar{r}$, and use the metric to find the distance between the points.

Chapter 30

30.2.1,2 *Hint*: You will need much less space than provided.

30.4.1 *Hint:* Ignore terms of order ξ^2, ξh, and $\xi^2 h$.

30.5.1 *Hint:* You will find that in an expression involving 12 terms, the 8 terms involving ξ^α cancel in pairs.

P30.3 (a) *Partial answer:* $B = -8\pi G\rho/3$. (b) *Partial answer:* $g_{tt} = -1 - Gm(r)/r + \frac{1}{2}C$. (c) *Partial answer:* $C = 6GM/R$.

P30.4 *Hint:* All but one of the $h_{\mu\nu}$ components turn out to be zero.

P30.5 (d) *Partial answer:* $g_{xx} = 1 + A\cos(\omega t - \omega z)$

P30.7 (f) *Hints*: You can assume that we can have $k_\beta A_\nu + k_\nu A_\beta = 0$ and yet have $A_{\mu\nu} \neq 0$. Also note that we can always orient spatial coordinates so that $k_x = k_y = 0$ (i.e. so that the z axis coincides with the $\vec{k}$ vector).

Chapter 31

31.1.1 First show that $\partial_\alpha(k_\sigma x^\sigma) = k_\alpha$.

31.2.2 Make sure that you understand why $k^\mu B_\mu = k_\nu B^\nu$.

31.2.3 *Partial answer:* $B^t = (A^{tt} + \frac{1}{2}A^\sigma{}_\sigma)/2\omega$.

P31.1 (c) *Partial answer:* $A'^{xx} = \frac{1}{2}(b - c)$.

P31.2 (c) *Partial answer:* $A'^{xy} = b$.

P31.4 *Hint:* Throw away terms that are of order A_+^2.

P31.6 (b) *Partial answer:* rotation rate: $\frac{1}{2}\omega$.

P31.7 *Hint:* A geodesic in this coordinate system is $x_g(t) = \pm\frac{1}{2}L$. Argue that if we instead keep the particle's distance s from the origin fixed, the particle will follow a path $x(t) \approx \pm\frac{1}{2}L[1 - \frac{1}{2}A_+\cos\omega t]$. So what is the particle's acceleration relative to the geodesic?

Chapter 32

32.3.1 *Hint:* First show that $h^{ij} = h_{ij}$ for spatial indices.

P32.1 *Hint:* In SI unit flux would have units of W/m^2.

P32.4 (a) *Partial answer:* the values are unchanged. *Hint:* Use the tensor transformation law and the Lorentz transformation coefficients. For spatial indices $h^{ij} = h_{ij}$ (as you can easily show). (b) *Hint:* Both the energy and the flux terms decrease by $(1-\beta)/(1+\beta)$.

P32.5 *Answer:* about 16 W/m^2.

P32.6 *Hint:* order of magnitude of fW/m^2.

P32.7 *Hint*: order of magnitude of tenths of fW/m^2.

P32.8 *Partial answer:* yes.

Chapter 33

33.5.1 *Hint:* This exercise requires a lot of work, but just keep at it. Note that $P^j_j = \delta^j_j - n^j n_j = 3 - 2 = 1$. Also note that $P_{jk} = \eta_{jk} - n_j n_k$, as you can easily show (and similarly for the raised-index version).

33.6.1 Don't try to evaluate the integrals from scratch: use Wolfram-Alpha or some other tool (just give credit).

P33.1 *Hint*: You will be able to see a merger anywhere in the universe.

P33.5 (a) *Partial answer:* $I\!\!\!-^{xx} = 2ML^2(\cos^2\theta - \frac{1}{3})$. (c) *Hint:* You should get $\frac{128}{5} \times$ (powers of G, M, L, ω).

P33.6 (a) *Partial answer:* $I\!\!\!-^{xx} = \frac{2}{3}ML^2$.

P33.7 (a) *Partial answer:* $I\!\!\!-^{xx} = \frac{4}{3}ML^2[L_0^2 + 2L_0 A\sin\omega t + A^2\sin^2\omega t]$.

P33.8 (c) *Partial answer:* The wave is uprightly polarized and has two frequency components.

P33.9 (b) *Partial answer:* $h^{xx}_{TT} = -(GML^2\omega^2/6R)[\cos(2\omega\{t - R\})]$. (c) *Hint:* order of magnitude of 10^{-33}/y.

Chapter 34

34.2.1 *Hint:* You will need much less space than provided.

P34.2 *Partial answer:* $dT/dt = -5.9 \times 10^{-13}$.

P34.3 *Partial answer:* $A_+ = 1.50 \times 10^{-21}$.

P34.4 *Partial answer:* $dT/dt = -1.23 \times 10^{-12}$.

P34.5 *Hint:* Don't be alarmed that the power you calculated is significantly less than the observed power. This is because the elliptical nature of the stars' orbits makes a big difference here.

P34.6 *Hint:* Order of magnitude of 100s of watts.

P34.7 *Hint:* You don't need to calculate actual luminosity values to rank them. Jupiter is anomalous.

P34.8 *Hint:* Order of 10^{15} y.

P34.9 (a) Partial answer: $I\!\!\!-^{xx} = \frac{4}{3}MA^2(\cos^2\omega t)$. (b) *Hint:* Don't forget the retarded time! (c) *Hint:* $-dE/dt = \frac{64}{15} \times$ (appropriate powers of G, m, A, ω).

P34.10 (c) *Hint:* Show that h_{TT}^{jk} is constant. (d) *Hint:* $\frac{112}{5} \times$ (appropriate powers of G, m, d, and v_0).

Chapter 35

35.2.1 *Hint:* When calculating $\vec{\nabla} \times \vec{B}_G$, add $0 = \partial^2 A_G/\partial t^2 - \partial^2 A_G/\partial t^2$ at an appropriate point.

35.3.1 *Hint:* Show that $h_{tj} = -h^{tj} = +4A_G^j$.

P35.1 *Hint:* Note that Γ^t_{tt} is not necessarily zero when the fields are time-dependent. *Answer:*

$$\vec{a} = E_G + 4\vec{v} \times \vec{B}_G - \vec{v}\frac{\partial \Phi_G}{\partial t} - 3\frac{\partial \vec{A}}{\partial t}.$$

P35.2 *Hint:* Remember that $\vec{B}_G$ obeys a *left*-hand rule with respect to its source, not a right-hand rule.

P35.3 *Hint:* Note that $h_{\mu\nu}$ must be unitless, because we add it to $\eta_{\mu\nu}$.

P35.5 *Partial answer:* The geodetic precession rate is 1.19 rad/s, which is the order of magnitude of 10 million times larger than the Lense-Thirring rate.

P35.6 (a) *Partial answer:* Yes. (b) *Hint:* A newtonian calculation is sufficient. *Partial answer:* the sideward acceleration depends on $(R/r)^{7/2}$.

P35.7 (e) *Hint:* s^ϕ is measured with respect to what direction?

Chapter 36

36.3.1 *Hint:* Simply make a list of the various terms, specifying for each where the volume element is located that cancels the volume element at *x, y, z* when you do the integral over all elements.

P36.1 (b) *Hint:* Note that r is not the same as R. Insert what r is in terms of R into $1 + 2GM/r$ and use the binomial approximation.

P36.2 *Hint:* The value of a/GM for the sun is order of magnitude of 1; the earth's value is much larger.

P36.3 *Hint:* The value of a/GM is somewhat smaller than 1.

P36.4 (b) *Partial answer:* r is not circumferential. (c) *Hint:* The distance is $> 2\pi r$. (d) *Partial answer:* Yes.

P36.5 *Answer:* $1.405GM$.

Chapter 37

37.1.1 *Hint:* To isolate the $dt/d\tau$ term, multiply the 1st equation by $g_{\phi\phi}$ and the 2nd equation by $g_{t\phi}$ and add. You can isolate the $d\phi/d\tau$ term similarly.

37.3.1 *Hint:* Note that $R^2\sin^2\theta \equiv ([g_{t\phi}]^2 - g_{tt}g_{\phi\phi})$ becomes simply R^2. on the equatorial plane. Multiply both sides of equation 37.18 by R^4, and *don't* yet substitute in the values of g_{tt}, $g_{t\phi}$, and $g_{\phi\phi}$, but *do* substitute in the values of $dt/d\tau$ and $d\phi/d\tau$ in terms of a, e, and g_{tt}, $g_{t\phi}$, and $g_{\phi\phi}$. You will be able to cancel a factor of R^2 from both sides. Then, and *only* then, substitute in the values of g_{tt}, $g_{t\phi}$, and $g_{\phi\phi}$ on the equatorial plane. If you substitute the values in too early, you will go crazy.

37.4.2 *Hint:* Note that $GMa^2 - r^3 = (\sqrt{GM}\,a + r^{3/2})(\sqrt{GM}\,a - r^{3/2})$.

P37.1 *Hint:* Argue that for a particle at rest, the metric equation implies that $-d\tau^2 = ds^2 = g_{tt}dt^2$.

P37.2 *Hint:* Show that $\ell = -(2GMa/r_0)u_0$.

P37.3 (a) *Partial answer:* $d\phi/d\tau = (5GM/R^2)(GM/r - 1/4)$.

P37.4 *Partial answer:* The radius is never smaller than $r = 2GM$.

P37.5 *Hint:* You should find that

$$\frac{d\phi}{d\tau} = \frac{B\sqrt{R_0^2\omega_0^2 + A_0} - (A/A_0)(B_0\sqrt{R_0^2\omega_0^2 + A_0} + R_0^2\omega_0}{R^2}$$

P37.6 (a) *Partial answer:* one of the periods is 1337 μs. (b) *Partial answer:* One of the periods is 1020 μs.

Chapter 38

38.2.2 *Hint:* Note that outside the event horizon, $R^2\sin^2\phi = [g_{t\phi}]^2 - g_{tt}g_{\phi\phi} > 0$.

P38.3 *Partial answer:* one of the speed limits will be $\sqrt{5/33}$.

P38.4 *Answer:* $1.977GM$.

P38.7 *Partial answer:* $v = 1$.

P38.9 (d) *Partial answer:* at $r = 2GM$, $v = 0.408$.

P38.10 (d) *Partial answer:* $(\boldsymbol{o}_\theta)^\theta = -1/\sqrt{g_{\theta\theta}}$, other components zero (minus so that the ZAMO's axes are right handed).

Chapter 39

39.4.1 *Hint:* Argue that $me = \delta M$ and $m\ell = \delta S$.

P39.2 (b) *Partial answer:* If it were to emit its energy as a rate equal to that of the entire galaxy, the black hole should be able radiate for about 700 million years.

P39.4 *Hint:* You should find that b blows up as we approach $r = 2GM$ in the equatorial plane.

P39.5 (d) *Partial answer:* The lower limit on v_ϕ is 0.389.

P39.6 (d) *Answer:* $0.586m$.

P39.7 (c) *Answer:* $T = \hbar\beta/[4\pi k_B GM(1+\beta)]$, where $\beta \equiv \sqrt{1-(a/GM)^2}$.

P39.8 (c) *Partial answer:* order of 10^{37} W.

Solutions to Supplementary Problems

S4.1 According to the definition of the dot product and the Lorentz transformation we have

$$\boldsymbol{A}' \cdot \boldsymbol{B}' = A'^{\mu}\eta_{\mu\nu}B'^{\nu} = (\Lambda^{\mu}{}_{\alpha}A^{\alpha})\eta_{\mu\nu}(\Lambda^{\nu}{}_{\beta}B^{\beta}) \tag{S4.1.1}$$

This describes a sum involving 4 × 4 × 4 × 4 = 256 terms, but in each term, the multiplication of the components is both commutative and associative, so we can write this as

$$\boldsymbol{A}' \cdot \boldsymbol{B}' = (\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta})A^{\alpha}B^{\beta} \tag{S4.1.2}$$

Now compare the quantity in parentheses to the right side of equation 4.18: you should see that it is the same. Therefore, we can replace that quantity with the left side of that equation, implying that

$$\boldsymbol{A}' \cdot \boldsymbol{B}' = \eta_{\alpha\beta}A^{\alpha}B^{\beta} \equiv \boldsymbol{A} \cdot \boldsymbol{B} \tag{S4.1.3}$$

This means that the numerical values of the two dot products is the same.

S12.1 **a.** The easiest way to get the answer to this part is to use Kepler's third law (equation 10.12)

$$\frac{4\pi^2}{\Omega^2} = \frac{4\pi^2 r_c^3}{GM} \quad\Rightarrow\quad \frac{d\phi}{dt} \equiv \Omega = \sqrt{\frac{GM}{r_c^3}} \quad\Rightarrow\quad v_\infty \equiv r\frac{d\phi}{dt} = r\Omega = \sqrt{\frac{GM}{r_c}} = \sqrt{\frac{GM}{6GM}} = \sqrt{\frac{1}{6}} \tag{S12.1.1}$$

b. The Schwarzschild metric implies that

$$d\tau^2 = -ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \frac{dr^2}{1 - 2GM/r} - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 \tag{S12.1.2}$$

For the case of a circular orbit in the equatorial plane, $dr = 0$, $\theta = \pi/2$, and $d\theta = 0$ for every pair of infinitesimally separated events along the path. If we divide the remaining terms in the above by dt and use the result of the last part, we get

$$\left(\frac{d\tau}{dt}\right)^2 = \left(1 - \frac{2GM}{r_c}\right) - \left(r_c\frac{d\phi}{dt}\right)^2 = \left(1 - \frac{2GM}{r_c}\right) - \frac{GM}{r_c} = 1 - \frac{3GM}{r_c} = 1 - \frac{3GM}{6GM} = \frac{1}{2} \tag{S12.1.3}$$

Therefore, $d\tau/dt = \sqrt{1/2}$, and $dt/d\tau = \sqrt{2}$, as claimed. Therefore

$$v_P \equiv r_c\frac{d\phi}{d\tau} = r_c\frac{d\phi}{dt}\frac{dt}{d\tau} = \sqrt{\frac{1}{6}}\sqrt{2} = \sqrt{\frac{1}{3}} \tag{S12.1.4}$$

This is larger than v_∞, as one might expect, because a clock that is both deeper in the gravitational well and moving should be running slower than a clock at rest at infinity,

c. The particle's four-velocity $\boldsymbol{u}_P$ has Schwarzschild components

$$\boldsymbol{u}_p = \begin{bmatrix} dt/d\tau \\ 0 \\ 0 \\ d\phi/d\tau \end{bmatrix} = \begin{bmatrix} (1 - 3GM/r)^{-1/2} \\ 0 \\ 0 \\ \sqrt{GM/r^3}\,(1 - 3GM/r)^{-1/2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ [6\sqrt{3}\,GM]^{-1} \end{bmatrix} \tag{S12.1.5}$$

Now, we can calculate the components of the particle's four-velocity in the falling observer's frame as follows: $u^t_{P(F)} = -\boldsymbol{o}_t \cdot \boldsymbol{u}_P$, $u^x_{P(F)} = \boldsymbol{o}_x \cdot \boldsymbol{u}_P$, $u^y_{P(F)} = \boldsymbol{o}_y \cdot \boldsymbol{u}_P$, and $u^z_{P(F)} = \boldsymbol{o}_z \cdot \boldsymbol{u}_P$, where, $\boldsymbol{o}_t$, $\boldsymbol{o}_x$, $\boldsymbol{o}_y$, and $\boldsymbol{o}_z$ are the observer's basis vectors, which (according to problem P12.7) have Schwarzschild components

$$\boldsymbol{o}_t = \begin{bmatrix} (1 - 2GM/r)^{-1} \\ -\sqrt{2GM/r} \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{o}_x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/r\sin\theta \end{bmatrix}, \quad \boldsymbol{o}_y = \begin{bmatrix} 0 \\ 0 \\ -1/r \\ 0 \end{bmatrix}, \quad \boldsymbol{o}_z = \begin{bmatrix} -\sqrt{2GM/r}\,(1 - 2GM/r)^{-1} \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{S12.1.6}$$

Again, note that in the equatorial plane, $\theta = \pi/2$. Since the Schwarzschild metric is diagonal, evaluating the components of the particle's four-velocity in the falling observer's frame yields

$$u^t_{P(F)} = -\boldsymbol{o}_t \cdot \boldsymbol{u}_P = -(\boldsymbol{o}_t)^{\mu}g_{\mu\nu}u^{\nu}_P = -(\boldsymbol{o}_t)^t g_{tt}u^t_P - (\boldsymbol{o}_t)^r g_{rr}u^r_P - (\boldsymbol{o}_t)^{\theta} g_{\theta\theta}u^{\theta}_P - (\boldsymbol{o}_t)^{\phi} g_{\phi\phi}u^{\phi}_P = -(\boldsymbol{o}_t)^t g_{tt}u^t_P$$

$$= -\left(1 - \frac{2GM}{r}\right)^{-1}\left[-\left(1 - \frac{2GM}{r}\right)\right]\left(1 - \frac{3GM}{r}\right)^{-1/2} = +\left(1 - \frac{3GM}{r}\right)^{-1/2} = \sqrt{2} \tag{S12.1.7a}$$

$$u^x_{P(F)} = \boldsymbol{o}_x \cdot \boldsymbol{u}_P = (\boldsymbol{o}_x)^\mu g_{\mu\nu} u^\nu_P = (\boldsymbol{o}_x)^t g_{tt} u^t_P + (\boldsymbol{o}_x)^r g_{rr} u^r_P + (\boldsymbol{o}_x)^\theta g_{\theta\theta} u^\theta_P + (\boldsymbol{o}_x)^\phi g_{\phi\phi} u^\phi_P = (\boldsymbol{o}_x)^\phi g_{\phi\phi} u^\phi_P$$

$$= \frac{1}{r} r^2 \sqrt{\frac{GM}{r^3}} = \sqrt{\frac{GM}{r}} = \sqrt{\frac{1}{6}} \tag{S12.1.7b}$$

$$u^y_{P(F)} = \boldsymbol{o}_y \cdot \boldsymbol{u}_P = (\boldsymbol{o}_y)^\mu g_{\mu\nu} u^\nu_P = (\boldsymbol{o}_y)^t g_{tt} u^t_P + (\boldsymbol{o}_y)^r g_{rr} u^r_P + (\boldsymbol{o}_y)^\theta g_{\theta\theta} u^\theta_P + (\boldsymbol{o}_y)^\phi g_{\phi\phi} u^\phi_P = 0 \tag{S12.1.7c}$$

$$u^z_{P(F)} = \boldsymbol{o}_z \cdot \boldsymbol{u}_P = (\boldsymbol{o}_z)^\mu g_{\mu\nu} u^\nu_P = (\boldsymbol{o}_z)^t g_{tt} u^t_P + (\boldsymbol{o}_z)^r g_{rr} u^r_P + (\boldsymbol{o}_z)^\theta g_{\theta\theta} u^\theta_P + (\boldsymbol{o}_z)^\phi g_{\phi\phi} u^\phi_P = (\boldsymbol{o}_z)^t g_{tt} u^t_P$$

$$= \frac{-\sqrt{2GM/r}}{\cancel{1 - 2GM/r}}\left[-\cancel{\left(1 - \frac{2GM}{r}\right)}\right]\left(1 - \frac{3GM}{r}\right)^{-1/2} = +\sqrt{\frac{1}{3}}\sqrt{2} = \sqrt{\frac{2}{3}} \tag{S12.1.7d}$$

So the components of the particle's ordinary velocity $\vec{v}_F$ measured in the freely-falling frame are

$$v_{Fx} = \frac{u^x_{P(F)}}{u^t_{P(F)}} = \sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}} = \sqrt{\frac{1}{6}}, \quad v_{Fy} = 0, \quad v_{Fz} = \frac{u^z_{P(F)}}{u^t_{P(F)}} = \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}} = \sqrt{\frac{1}{3}} \tag{S12.1.8a}$$

and the particle's speed in this frame is

$$v_F = \sqrt{v_{Fx}^2 + v_{Fy}^2 + v_{Fz}^2} = \sqrt{\frac{1}{6} + \frac{1}{3}} = \sqrt{\frac{1}{2}} \tag{S12.1.8b}$$