A GENERAL RELATIVITY WORKBOOK

A GENERAL RELATIVITY WORKBOOK

Thomas A. Moore
Pomona College

University Science Books
Mill Valley, California

University Science Books
www.uscibooks.com

Production Manager: Paul Anagnostopoulos
Text Design: Yvonne Tsang
Cover Deisign: Genette Itoko McGrew
Manuscript Editor: Lee Young
Illustrator: Laurel Muller
Compositor: Cohographics
Printer & Binder: Victor Graphics

ISBN 978-1-891389-82-5

This book is printed on acid-free paper. ♾

Library of Congress Cataloging-in-Publication Data

Moore, Thomas A. (Thomas Andrew)
A general relativity workbook / Thomas A. Moore, Pomona College.
pages cm
Includes index.
ISBN 978-1-891389-82-5 (alk. paper)
1. General relativity (Physics) I. Title.
QC173.6.M66 2012
530.11—dc23
2012025909

Printed in North America
10 9 8 7 6 5 4 3 2 1

For Joyce, whose miraculous love always supports me and
allows me to take risks with life that I could not face alone,

and for Edwin Taylor, whose book with Wheeler set me on this path decades ago,
and whose gracious support and friendship has kept me going.

CONTENTS

20. THE STRESS-ENERGY TENSOR 231

PREFACE

Introductory Comments. General relativity is one of the greatest triumphs of the human mind. Together with quantum field theory, general relativity lies at the foundation of contemporary physics and currently represents the most durable physical theory in existence, having survived nearly a century of development and increasingly rigorous testing without being contradicted or superseded. Long admired for its elegant beauty, general relativity has also (particularly in the past two decades) become an essential tool for working physicists. It provides the basis for understanding a huge variety of astrophysical phenomena ranging from active galactic nuclei, quasars, and pulsars to the formation, characteristics, and destiny of the universe itself. It has driven the development of new experimental tools for testing the theory and for the detection of gravitational waves that represent one of the most lively and challenging areas of contemporary physics. Even engineers are starting to have to pay attention to general relativity: making the Global Positioning System function correctly requires careful attention to general relativistic effects.

In some ways, general relativity was so far ahead of its time that it took a long time for instrumentation and applications to catch up sufficiently to make it more than than an intellectual adventure for the curious. However, as general relativity has now moved firmly into the mainstream of contemporary physics with a wide and growing variety of applications, teaching general relativity to undergraduate physics majors has become both relevant and important, and the need for appropriate and up-to-date undergraduate-level textbooks has become urgent.

Audience. This textbook seeks to support a one-semester introduction to general relativity for junior and/or senior undergraduates. It assumes only that students have taken multivariable calculus and some intermediate Newtonian mechanics beyond a standard treatment of mechanics and electricity and magnetism at the introductory level (though students who have also taken linear algebra, differential equations, some electrodynamics, and/or some special relativity will be able to move through the book more quickly and easily). This book has grown out of my experience teaching fourteen iterations of such an undergraduate course during my teaching career.

Those iterations have convinced me not only that undergraduates *can* develop a solid proficiency with the general relativity, but also that studying general relativity provides a superb introduction to the best practices of theoretical physics as well as a uniquely exciting and engaging introduction to ideas at the very frontier of physics, things that students rarely experience in other undergraduate courses.

Pedagogical Principles. Since students rarely see the tensor calculus used in general relativity in undergraduate mathematics courses, a course in general relativity must either teach this mathematics from scratch or seek to work around it (at some cost in coherence and depth of insight). In my experience, junior and senior undergraduates *can* master tensor calculus in an appropriately designed course, and that doing this is well worth the effort, as it provides the firm foundation needed for confidence and flexibility in confronting applications.

The pedagogical key for developing this mastery is for you (the student) to *personally own* the mathematics by working through most of the arguments and derivations *yourself.* Therefore, I have designed this textbook as a *workbook*. Each chapter opens with a concise core-concept presentation that helps you see the big picture without mathematical distraction. This presentation is keyed to subsequent "boxes" that I have designed to guide you in working through the supporting derivations as well as other details and applications whose direct presentation would obscure the core ideas. I have found this combination of overview and guided effort to be uniquely effective in building a practical understanding of the theory's core concepts and their mathematical foundations.

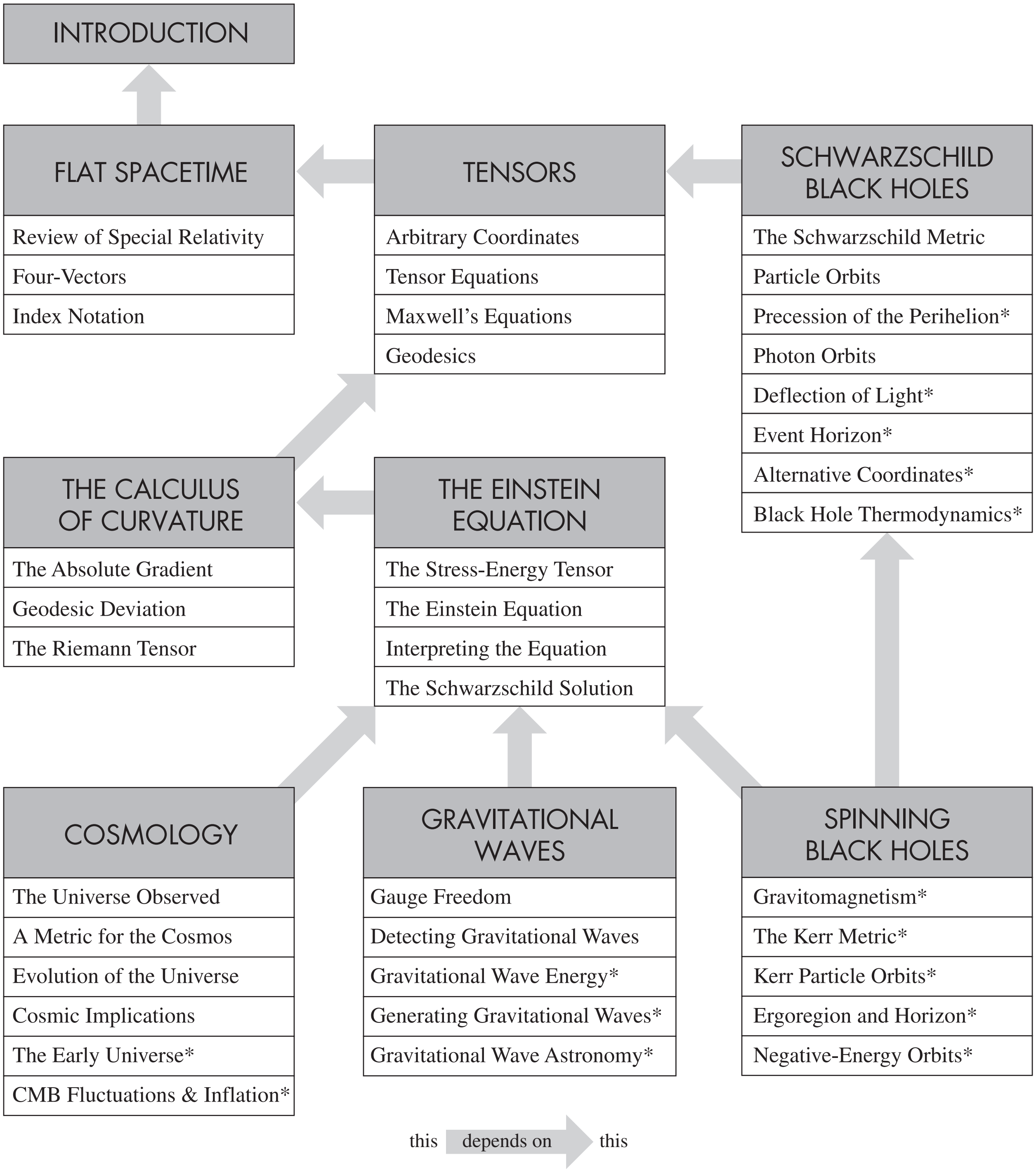
The overview-and-box design also helps keep you focused on the *physics* as opposed to the mathematics, underlining how the mathematics *supports* and *expresses* the physics. Other aspects of the textbook's design also support the principle that the physics should be foremost. I have ordered the topics so that the mathematics is presented not in one big lump but rather gradually and "as needed," thus allowing the physics to drive the presentation. For example, you will extensively practice using tensor notation by exploring real physical applications in flat space before learning about the geodesic equation that describes an object's motion in a curved spacetime. You will then spend a great deal of time exploring the physical implications of the geodesic equation in the particular curved spacetime surrounding a simple spherical object before learning the additional mathematical tools required to show *why* spacetime is curved in that particular way around a spherical object. Along the way, I use many "toy" examples in two-dimensional flat and curved spaces help develop your intuitive understanding of the physical meaning of the core ideas. The gradual development of the mathematics throughout the text also helps ensure that you have time to gain a firm footing for each step before continuing the climb.

The key to using this book successfully is working carefully through all the boxes in this book. Doing this will ultimately provide you with a range of experience and depth of understanding difficult to obtain any other way.

Chapter Dependencies. The chart that appears on each chapter's title page (and on the next page) shows how the major sections of the book depend on each other. For example, you can see from the chart that the **Introduction**, **Flat Space**, and **Tensors** sections (chapters 1 through 8) provide core material that every other section uses. After chapter 8, I strongly recommend going on to the **Schwarzschild Black Holes** section, because this will develop your understanding of how to work with curved spacetimes before having to wrestle with yet more math (and because black holes are fascinating applications of the theory). However, this is not essential; in a short course focused on cosmology, for example, one could go directly on to the **Calculus of Curvature**, **Einstein Equation**, and **Cosmology** sections. Note also that the final three sections (**Cosmology**, **Gravitational Waves**, and **Spinning Black Holes**) are completely independent of each other and can be explored in any order one might choose. However, all three of these sections require the **Calculus of Curvature** and **Einstein Equation** sections.

One also does not have to go all the way through the Schwarzschild section. The last three chapters (on black holes) are only necessary if you also plan to go through the last two chapters of the **Spinning Black Hole**s section (though it is hard to imagine why anyone would want to avoid learning about black holes!). One can easily omit the *Deflection of Light* chapter without loss of continuity. The *Precession of the Perihelion* chapter is necessary background for the *Deflection of Light* chapter, but you could omit both. The first two chapters are required for all of the other chapters in this section, and the fourth chapter on *Photon Orbits* presents a mathematical technique that is employed in certain homework problems throughout the rest of the book, but it is only absolutely required for the *Deflection of Light* and the *Black Hole Thermodynamics* chapters.

In the **Cosmology** section, the first four chapters provide core material and should all be included if this section is to be explored at all. The last two chapters, however, are completely optional; you can omit either both or the last, as desired.



A chart showing the chapters of the book grouped in their major sections and how those sections depend on each other. Chapters marked with a * are optional, though later optional chapters typically depend on earlier such chapters.

While in principle it is possible to stop at after the first two chapters in the **Gravitational Wave** section, I think that a discussion of gravitational wave energy and generation is pretty important. I therefore recommend going through at least the first three chapters of this section if you want to explore gravitational waves at all.

One might reasonably elect to explore the *Gravitomagnetism* chapter alone in the **Spinning Black Holes** section, or stop after either the *Kerr Particle Orbits* chapter or the *Ergoregion and Horizon* chapter. However, the chapters in this section *do* need to be discussed in sequence; one cannot easily drop one from the middle.

The First Chapter. Please also note that the *first* chapter has a different structure than the others. After dealing with preliminaries, I usually end the first class session of the course I teach with a 40-minute interactive lecture. For the sake of completeness (and for later reference), I have provided in the first chapter what amounts to a polished transcript of that lecture. This chapter has no boxes because I don't expect my students to have read (or perhaps even own) the book before the first class. To help them track the lecture, I instead give them the two-sided handout that appears as the last two pages of the first chapter.

The Second Chapter. This chapter presents a very terse review of special relativity aimed primarily at students that have already encountered some relativity in a previous course. If you have not seen relativity before, you may find this chapter harder going. Even so, everything you need to know about special relativity for this book is presented there, and if you work through the chapter slowly, and do many of the homework problems, you should be fine. I have also included references to supplemental reading that you may find helpful.

Book Website. You can find a variety of other helpful information and supporting computer software on this textbook's website:

http://pages.pomona.edu/~tmoore/grw/

Please also feel free to email me suggestions, questions, and error notices: my email address is tmoore@pomona.edu.

Information for Instructors. So far in this preface, I have addressed issues of concern to all readers in language directed mostly to students. In the remainder, I want to specifically address issues of interest to instructors who are designing undergraduate courses around this book.

Course Pacing. I have designed the text so that (in my experience) *each chapter can generally be discussed in a single* (50-minute) *class session*, particularly if you use the format for class sessions I describe below. Your mileage may vary (for example, you may need to spend more time on chapter 2 if your students' background in special relativity is weak), but this general rule should help you appropriately pace the course.

You also have a lot flexibility in choosing which chapters to cover and which you might omit: there are at least twenty different chapter sequences that make sense. Be sure to examine thoroughly the section above on **Chapter Dependencies** before designing a syllabus that omits chapters. However, I find that I can usually get through the entire book in one semester.

Let me emphasize again that the last three sections (**Cosmology**, G**ravitational Waves**, and **Spinning Black Holes**) are independent; you can present them in any order. One of my colleagues likes to end the course with cosmology, which he thinks provides an exciting climax. I have made that section first of the three precisely because I *also* think it is the most important. If I am working through the book sequentially and run out of time, I'd rather do so in the Spinning Black Holes section than omit any of the cosmology material! I also find that students have many other pressures and concerns near the end of the semester, so I tend to schedule material that I consider *less* crucial toward the end. But you can certainly choose what works best for you and your students, and you have lots of flexibility to do so.

How to Spend Class Time. The workbook format will push students to gain mastery *only* if your course design somehow rewards students for filling out the boxes. The last time I taught the course, I asked several students chosen at random each class session to hand me their books, which I subsequently graded for thoughtful *effort* in filling out the boxes since the last time they submitted their book, with special emphasis on the chapter discussed in class that day. Each student's average grade for these random samples counted about 13% of their course grade. I arranged things so that each student was called on about five to six times a semester.

One of my colleagues at a another institution uses a different approach that may be even better. After determining which box exercises seemed easy enough to skip discussion, he then assigns each remaining box exercise to a student in a strict rotation (including himself in the rotation). The student must present the solution in front of the class. This strongly motivates the students to come to class prepared without having to assign a formal grade for preparation, and also makes class time a bit more active than the way I did it. I intend to use this approach myself the next time that I teach the course.

You might find some other approach better than either of these for your students, but I consider it very important when designing a course based on this book to find *some* way of rewarding students for doing work in the boxes before class.

In either of the approaches outlined above, we spend much of the class period discussing the challenges students encountered in going through the boxes. Because students have at least *tried* to work out the boxes before class, they typically bring good questions to the table, questions that directly address the difficulties they are experiencing personally. We are therefore able to spend class time efficiently addressing students' *actual needs*. If we have time (and we often do), I often work some example problems in class, targeted toward either some interesting physics and/or preparing them better to do the homework. In my experience, this approach to using class time is much more effective and efficient than lecturing would be.

I also recommend that you (the instructor) work through all the boxes in an assigned chapter *yourself* before class. (I myself do this every time I offer the course, even though I have worked through all the boxes several times now!) This will help refresh your memory, help isolate any issues that you might need to resolve for yourself before class, and (most importantly) help you anticipate and appreciate the difficulties that students will have with the boxes.

I intentionally designed most of the boxes so that they ask students to prove something, as the primary goal of the boxes is to help students gain ownership of the concepts and derivations discussed in the text. The homework problems are usually much more open-ended, providing opportunities for students to extend the ideas presented in the text, explore physical applications, and even think about new topics. Some of the problems are also designed to provide a basis for class discussion of topics not covered in the main text.

Homework. I typically assign about two homework problems per chapter: this is enough to keep students pretty busy. Homework problems for this class can be pretty challenging, and even the best students may not get them right the first time. Homework-grading schemes that focus *only* on the final results can therefore make students anxious. However, one can devise grading schemes that (1) allow students to engage difficult problems without anxiety, (2) provide them with an opportunity for further learning, and (3) make grading easier for you or your TAs. The "Course Design" section of the book's website provides a link to a page that discusses a scheme for grading homework that I strongly recommend that you consider: it not only encourages students to tackle tough problems without fearing failure, but I can also guarantee that it will save you time grading!

Website for Instructors. I have set up a special, controlled-access website especially for instructors. If you are an instructor that has adopted the book, send me (1) your name, (2) your institution, and (3) how many students are in your course, and I will tell you how to access that site. This site includes complete problem solutions, box solutions, sample tests, and other information that instructors will find helpful but which should not be available to students in an uncontrolled way.

I also welcome emails if you have questions, error notices, or other comments.

Appreciation. I am grateful to many people have helped bring this text to fruition. First, let me thank the students in my Physics 160 class (and particularly Nathan Reed and Ian Frank) for documenting errors in early versions and offering feedback. The overview/box idea grew out of conversations with Dayton Jones (more than thirty years ago). A fruitful correspondence with James Hartle helped me formulate my goals for this book, and his excellent text both taught and inspired me. I am grateful to Edwin Taylor, who has challenged me and broadened my perspective since I was a high school student, for personal support and insight. Thanks also to Tom Baumgarte and Ben Sugerman for trying initial versions of the book in their classes and offering thoughtful feedback. I am grateful to Tom Helliwell, Nandor Bokor, Nelson Christensen and his students (Tom Callister, Ross Cawthon, Andrew Chael, Micah Koller, Dustin Anderson, and David Miller, who all sent me individual reviews), Tom Baumgarte, Tom Carroll, Bryan van der Ende, and an unknown reviewer for reading a nearly final draft and offering a number of valuable suggestions and error corrections. Needless to say, any remaining errors are my own. I want to thank the developers of MathMagic (my equation-editing software) for extraordinary attention and help beyond the call of duty when I encountered various problems. Hilda Dinolfo and Christine Maynard were very helpful in printing copies for various early readers. I am very grateful to Sergio Picozzi and John Mallinckrodt for carefully reviewing the final draft and being willing to write such nice endorsements for the back cover. I want to thank the book's production team (Lee Young, Richard Camp, Yvonne Tsang, Genette Itako McGrew, and especially Laurel Muller and Paul Anagnostopoulos) for their excellent work, care, and extraordinary patience in dealing with a difficult book (and a sometimes difficult author). I also want to thank Jane Ellis, my managing editor at University Science Books, for her support, enthusiasm, and hard work in bringing this book to print, and for being willing to take a risk on a book that was a bit out of the ordinary.

Finally, let me thank my wife, Joyce, whose unfailing support for my writing habit is loving and gracious beyond the call of duty. I am very grateful to all!

Thomas A. Moore
Claremont, CA
July 11, 2012